THE PROBLEM OF KLEIN*

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One of the most important discoveries of analytical mechanics is the optical-mechanical analogue revealed (discovered) by Hamilton.

Hamilton established the likeness of the canonical forms

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

of the basic equations of the dynamics of conservative systems and of the wave theory of light of Huygens.

Discoveries of Hamilton, Jacobi, Poincaré and Helmholtz are closely connected with the optical-mechanical analogue. All this is well known.

Optical theory did not cease to develop after the wave theory of Huygens. There followed the theories of Fresnel, Cauchy and Maxwell.

Cauchy, who set himself as a goal the further development of the optical-mechanical analogue of Hamilton, found this analogy not in the area of the dynamics of systems of material points, but in the field of the theory of oscillations of an elastic medium. With his discovery, Cauchy diverted from analytical dynamics problems concerning the further development of the analogue with the post-Huygens theory of light.

Felix Klein was, probably, the first who directed his attention to this matter. It is for this reason that we shall refer to the problem of the further development of the optical-mechanical analogue within the framework of analytical dynamics, as the problem of Klein.

In order to solve the problem of Klein, we note that in all post-Huygens theories, light is considered as some oscillating process. The development of the optical-mechanical analogue should, therefore, be sought in the area of oscillatory motions.

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* The work was found in notes by N.G. Chetaev and was dated 1941.

Lagrange established that near the position of equilibrium of some mechanical systems there arise oscillations under small disturbances of the initial values of the coordinates q and of the momenta p when the position of equilibrium is stable. Conversely, if the position of equilibrium is stable, then in its neighborhood, any disturbed motion will have an oscillatory nature, and if they are of a destructive nature, then this is due to secular terms. For the case of periodic motion, the theorem of Lagrange was generalized by Poincaré and Liapunov independently of each other. They showed that if a certain periodic motion of a conservative system is stable, then the corresponding equations in Poincaré's variations will have solutions with zero characteristic numbers of Liapunov. I have been able to generalize the theorem of Poincaré and Liapunov to the general case of stable motions of conservative systems.

Thus, if the solution of Klein's problem exists, then one should seek it among the properties of stable motions of conservative systems.

1. On a property of stable motions of conservative systems. For the sake of simplicity let us assume that we have a holonomic system with one degree of freedom which is acted upon by a system of forces which admit a force function; let q be Lagrange's coordinate, p its momentum, and let H be Hamilton's function. The equations of motion have the form

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

The equations in variations of Poincaré for some leading or undisturbed motion will be

$$\frac{d\xi}{dt} = \frac{\partial^2 H}{\partial p \partial q}\xi + \frac{\partial^2 H}{\partial p^2}\eta, \qquad \frac{d\eta}{dt} = -\frac{\partial^2 H}{\partial q^2}\xi - \frac{\partial^2 H}{\partial q \partial p}\eta$$

where ξ, η denote the variations of the coordinate q and of the momentum p, respectively.

Let us assume that the leading motion is reducible and stable in the sense of Liapunov. For small initial disturbances ξ_0, η_0, the equations of Poincaré will always be equations of the first approximation. The solutions of the equations in the variations of Poincaré will have the form

$$\xi = \alpha\xi_0 + \beta\eta_0 \qquad \eta = \gamma\xi_0 + \delta\eta_0 \tag{1.1}$$

For two arbitrary solutions ξ, η and ξ', η' of the equations in the variations, Poincaré established the now well known invariant

$$\xi\eta' - \eta\xi'$$

In accordance with Poincaré's invariant we have

$$\begin{vmatrix} \xi & \eta \\ \xi' & \eta' \end{vmatrix} = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \cdot \begin{vmatrix} \xi_0 & \eta_0 \\ \xi_0' & \eta_0' \end{vmatrix} = \begin{vmatrix} \xi_0 & \eta_0 \\ \xi_0' & \eta_0' \end{vmatrix}, \text{ or } \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = 1$$

In other words, for every instant t, the transformations (1.1) represent a uni-modular group of linear transformations. If the leading motion is stable and reducible, then, in accord with the basic results of Liapunov's theory of stability, the equations of variation of Poincaré possess an invariant quadratic form of a definite sign, and the invariant quadratic form will be of the type

$$\xi\bar{\xi} + \eta\bar{\eta}$$

What are the interrelations which the invariance of this form imposes on the coefficients of the uni-modular linear transformation (1.1)?

We have

$$\xi\bar{\xi} + \eta\bar{\eta} = (\alpha\xi_0 + \beta\eta_0)\,(\bar{\alpha}\bar{\xi}_0 + \bar{\beta}\bar{\eta}_0) + (\gamma\xi_0 + \delta\eta_0)\,(\bar{\gamma}\bar{\xi}_0 + \bar{\delta}\bar{\eta}_0) =$$
$$= \alpha\bar{\alpha}\xi_0\bar{\xi}_0 + \alpha\bar{\beta}\xi_0\bar{\eta}_0 + \beta\bar{\alpha}\eta_0\bar{\xi}_0 + \beta\bar{\beta}\eta_0\bar{\eta}_0 + \gamma\bar{\gamma}\xi_0\bar{\xi}_0 + \gamma\bar{\delta}\xi_0\bar{\eta}_0 + \delta\bar{\gamma}\eta_0\bar{\xi}_0' + \delta\bar{\delta}\eta_0\bar{\eta}_0 =$$
$$= (\alpha\bar{\alpha} + \gamma\bar{\gamma})\,\xi_0\bar{\xi}_0 + (\alpha\bar{\beta} + \gamma\bar{\delta})\,\xi_0\bar{\eta}_0 + (\beta\bar{\alpha} + \delta\bar{\gamma})\,\eta_0\bar{\xi}_0 + (\beta\bar{\beta} + \delta\bar{\delta})\,\eta_0\bar{\eta}_0 = \xi_0\bar{\xi}_0 + \eta_0\bar{\eta}_0$$

Since this relation must hold for arbitrary initial values ξ_0, η_0, the following equations have to be valid

$$\alpha\bar{\alpha} + \gamma\bar{\gamma} = 1, \qquad \alpha\bar{\beta} + \gamma\bar{\delta} = 0$$
$$\beta\bar{\alpha} + \delta\bar{\gamma} = 0, \qquad \beta\bar{\beta} + \delta\bar{\delta} = 1$$

From the first column of the obtained relations we obtain

$$\bar{\alpha} = \begin{vmatrix} 1 & \gamma \\ 0 & \delta \end{vmatrix} = \delta, \qquad \bar{\gamma} = \begin{vmatrix} \alpha & 1 \\ \beta & 0 \end{vmatrix} = -\beta$$

while from the second column we have

$$\bar{\beta} = \begin{vmatrix} 0 & \gamma \\ 1 & \delta \end{vmatrix} = -\gamma, \qquad \bar{\delta} = \begin{vmatrix} \alpha & 0 \\ \beta & 1 \end{vmatrix} = \alpha$$

Thus, the matrix of the uni-modular transformation (1.1) has the following property

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \bar{\delta} & -\bar{\gamma} \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \tag{1.2}$$

in case the leading motion is stable also in the reduced variables ξ, η.

2. Representation of the obtained group of transformations. The revelation of an analogue between phenomena consists of showing the coincidence of the group of transformations of one phenomenon with the group of transformations of the other phenomenon. In case both groups turn out to be the same, then there exists an analogue between the two phenomena.

The essence of the analogue discovered by Hamilton lies in the fact that the group of transformations of the conservative systems, and the group of the propagation of light by Huygens' wave theory are groups of contact or canonical transformations. That the group of transformations of the dynamics of conservative systems is the group of tangent transformations was established by Hamilton in the basic property of the function of action V

$$\delta V = \sum p\delta q - \sum p^\circ \delta q^\circ$$

This asserts that the values q, p, and q°, p° are connected by the formulas of the contact transformations

$$p = \frac{\partial V}{\partial q}, \qquad -p^\circ = \frac{\partial V}{\partial q^\circ}$$

Hence, in order to find phenomena which are analogous to the disturbed motion of canonical systems in the neighborhood of a stable leading motion, it is necessary to consider representations of the obtained group of the uni-modular, linear transformations (1.1) with the property (1.2).

It has been shown quite clearly that the group of uni-modular linear transformations (1.1) has a representation in the proper Lorentz group. It is necessary to explain the nature of the property (1.2).

Let us consider the space ξ, η whose metric properties are determined by Poincaré's invariant

$$\xi\eta' - \eta\xi'$$

We obtain the so-called spinor space with the skewsymmetric fundamental tensor

$$\{g_{\alpha\beta}\} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

if we interpret the variations ξ, η as contravariant components of the vector ξ^1, ξ^2. The covariant components are determined by the usual formulas

$$\xi_\alpha = \sum g_{\alpha\beta}\xi^\beta$$

whence,

$$\xi_1 = \xi^2, \qquad \xi_2 = -\xi^1$$

If the transformations of the contravariant components happen to be the transformations (1.1), then we will have the following formulas for the covariant components ξ_1, ξ_2:

$$\xi_1 = \delta\xi_{01} - \gamma\xi_{02}, \qquad \xi_2 = -\beta\xi_{01} + \alpha\xi_{02}$$

From this it follows that the conjugate components $\bar{\xi}_1$, $\bar{\xi}_2$ will be transformed according to the formulas

$$\xi_1 = \bar{\bar{\delta}}\xi_{01} - \bar{\bar{\gamma}}\xi_{02}, \qquad \bar{\xi}_2 = -\bar{\bar{\beta}}\xi_{01} + \bar{\bar{\alpha}}\xi_{02}$$

If the leading motion is stable, that is if the relation (1.2) holds, then one obtains directly the formulas for the transformation

$$\bar{\xi}_1 = \alpha\bar{\xi}_{01} + \beta\bar{\xi}_{02}, \qquad \bar{\xi}_2 = \gamma\bar{\xi}_{01} + \delta\bar{\xi}_{02}$$

In other words, the components $\bar{\xi}_1$, $\bar{\xi}_2$ are transformed, in accord with the formulas (1.1) of the transformation of the contravariant components of the vector of the spinor space.

Thus, if the property (1.2) holds for an arbitrary vector (ξ), one may look upon the components $\bar{\xi}_1$, $\bar{\xi}_2$ as upon contravariant components η^1, η^2 of some corresponding vector (η). We shall express this circumstance analytically by following relations:

$$S\bar{\xi}_\alpha = \eta^\alpha, \qquad S\eta^\alpha = \bar{\xi}_\alpha \qquad (\alpha = 1, 2) \tag{2.1}$$

Van der Waerden has made a thorough study of the representations of the transformation S in his work "The method of group theory in quantum mechanics". He established that the transformation S augments with a reflection the representation of the uni-modular group of the linear transformations (1.1) in the proper group of Lorentz.

For the sake of completeness, I shall give here the method of such a representation. Let us consider the spin-tensor of the second order $\|c_{\mu\dot{\nu}}\|$ whose component $c_{\mu\dot{\nu}}$ is transformed as the product $\xi_\mu\bar{\xi}_\nu$. Such a tensor has the following invariant

$$C = \begin{vmatrix} c_{1\dot{1}} & c_{1\dot{2}} \\ c_{2\dot{1}} & c_{2\dot{1}} \end{vmatrix} = c_{1\dot{1}}c_{2\dot{2}} - c_{1\dot{2}}c_{2\dot{1}} = \tfrac{1}{2}\left(c_{1\dot{1}}c^{1\dot{1}} + c_{2\dot{2}}c^{2\dot{2}} + c_{1\dot{2}}c^{1\dot{2}} + c_{2\dot{1}}c^{2\dot{1}}\right) = \tfrac{1}{2}\sum c_{\mu\dot{\nu}}\, c^{\mu\dot{\nu}}$$

Let us consider the tensor $\|c_{\mu\dot{\nu}}\|$ for which the invariant quadratic form

$$\sum c_{\mu\dot{\nu}}\xi^\mu\bar{\xi}^\nu$$

takes on only real values. Here $c_{1\dot{1}}$, $c_{2\dot{2}}$ must be real, and $c_{1\dot{2}}$ and $c_{2\dot{1}}$ are complex conjugates. If we now introduce in place of $c_{\mu\dot{\nu}}$ new real variables x_0, x_1, x_2, x_3 in accordance with the relations

$$\begin{aligned} c_{1\dot{1}} &= x_3 + x_0, & c_{1\dot{2}} &= x_1 + ix_2 \\ c_{2\dot{1}} &= x_1 - ix_2, & c_{2\dot{2}} &= -x_3 + x_0 \end{aligned} \tag{2.2}$$

then the invariant C will take the following form in terms of the new variables:

$$C = \begin{vmatrix} x_3 + x_0 & x_1 + ix_2 \\ x_1 - ix_2 & -x_3 + x_0 \end{vmatrix} = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

This shows that under the transformation (1.1) the real variables x_k will be transformed in such a way as to leave invariant the expression $x_0^2 - x_1^2 - x_2^2 - x_3^2$, i.e., the variables x_k will undergo a real, proper Lorentz transformation.

Let us now apply the transformation S, given by formulas (2.1), to the tensor $\|c_{\mu\dot{\nu}}\|$

$$S\,\|c_{\mu\dot{\nu}}\| = \|c'^{\mu\dot{\nu}}\|$$

or, more explicitly

$$Sc_{1\dot{1}} = c'^{1\dot{1}} = c_{2\dot{2}}', \qquad Sc_{1\dot{2}} = c'^{2\dot{1}} = -c_{1\dot{2}}'$$
$$Sc_{2\dot{1}} = c'^{2\dot{1}} = -c_{1\dot{2}}' \qquad Sc_{2\dot{2}} = c'^{2\dot{2}} = c_{1\dot{1}}'$$

This, however, by formula (2.2) yields directly

$$x_0' = x_0, \quad x_1' = -x_1, \quad x_2' = -x_2, \quad x_3' = -x_3$$

Hence, the complete group of Lorentz transformations is a representation of the group of uni-modular linear transformation (1.1) with the property (1.2).

The complete Lorentz group is basic for the theory of light of Cauchy and Maxwell which appeared after Huygens' theory. One can look upon the obtained result as a solution of Klein's problem.

3. Another proof. It is reasonable to require another proof for the basic result.

In the first proof the property (1.2) of the reduced variables was used for stable disturbed motions. Let ξ^1, ξ^2 represent general variations of the coordinate and the impulse. According to the Poincaré invariant the components ξ^a are transformed as the group of uni-modular linear transformations

$$\xi^1 = \alpha\xi_0^1 + \beta\xi_0^2 \qquad \xi^2 = \gamma\xi_0^1 + \delta\xi_0^2 \qquad (\alpha\delta - \beta\gamma = 1) \tag{3.1}$$

Let the leading motion be stable, and the corresponding differential equations in the variations of Poincaré be reducible. Then, according to the results of Liapunov, there exists a definite (in the sense of Liapunov) real, invariant quadratic form known as Liapunov's function. In the spinor space of the group of uni-modular linear transformations (3.1) this function of Liapunov will be a quadratic form of the Hermite type

$$\sum c_{\mu\dot{\nu}}\xi^\mu\bar{\xi}^\nu \qquad (c_{\mu\dot{\nu}} = \overline{c_{\nu\dot{\mu}}})$$

where $|c_{\mu\dot{\nu}}|$ represents a bounded function of time. The criteria for the definiteness of a Hermitian form are well known

$$C = \begin{vmatrix} c_{1\dot{1}} & c_{1\dot{2}} \\ c_{2\dot{1}} & c_{2\dot{2}} \end{vmatrix} > 0 \qquad (c_{1\dot{1}} \neq 0) \tag{3.2}$$

Indeed, suppose that

$$\varphi^\alpha = \sum A^\alpha_\beta \, \xi^\beta$$

is a transformation to the canonical reduced variables ϕ^a in terms of which the equations of variations of Poincaré have constant coefficients. In accordance with the definition of reduced systems (Liapunov, *Obshchaia zadacha* [*General problem*] page 43) the coefficients $A_\beta{}^a$ satisfy the following conditions: they are continuous and bounded functions of time t; their first derivatives are functions of the same type, and the quantity

$$\frac{1}{\| A^\alpha_\beta \|}$$

is a bounded function of t. From this it follows that the now obvious Liapunov function of the canonical, reduced variables ϕ^a has the form

$$\sum \varphi^\alpha \bar{\varphi}^\alpha = \sum A^\alpha_\mu \, \bar{A}^\alpha_\nu \xi^\mu \bar{\xi}^\nu$$

For the stable disturbed motions, determined by the canonical equations in variations, the function of Liapunov is an invariant (a pair of pure imaginary characteristic roots). Hence,

$$c_{\mu\dot{\nu}} = \sum_\alpha A^\alpha_\mu \, \bar{A}^\alpha_\nu$$

are the components of a spin-tensor of the second order. The Hermitian nature of the quadratic form can be proved directly from the relation

$$c_{\mu\dot{\nu}} = \sum A^\alpha_\mu \, \overline{A^\alpha_\nu} = \overline{\sum \overline{A^\alpha_\mu} \;\; A^\alpha_\nu} = \overline{c_{\nu\dot{\mu}}}$$

The expressions $c_{\mu\dot{\nu}}$ are bounded, and the $c_{\mu\dot{\mu}}$ are positive.

We have seen that the discriminant of the Hermitian form C is an invariant of the group of the uni-modular linear transformations (3.1). The criterion for definiteness of (3.2) in the variables x_k takes for form

$$x_0{}^2 - x_1{}^2 - x_2{}^2 - x_3{}^2 > 0$$

The rest follows easily. Let us consider, therefore, the invariant

$$x_0{}^2 - x_1{}^2 - x_2{}^2 - x_3{}^2 = 1 > 0$$

In the space (Euclidean) $x_0 \; x_1 \; x_2 \; x_3$, this equation determines a two-sheeted hyperboloid. By geometrical considerations, like those that were developed by Poincaré in his work "On the basic hypotheses of geometry", one can represent a two-sheeted hyperboloid in a real three-dimensional space of Lobachevskii. To a definite set of values of the variables x_k there corresponds a point of the upper $x_0 > 0$ sheet of the two-sheeted hyperboloid, and, hence, also a point of the real part of the space of Lobachevskii. This representation can also be realized by the methods of Cayley, Cartan and others.

The totality of the transformations of the real Lobachevskii space represents, as is known, the complete Lorentz group, i.e. the basic group of transformations of the mathematical theory of light of Cauchy and of the electromagnetic theory of light of Maxwell. This proves that the group of the uni-modular linear transformations (3.1) for the case of a stable leading motion (if the equations in variation are reducible) has as its representation the complete Lorentz group.

The set of all linear transformations of the variables x_k in the real Lobachevskii space consists of the motions of the space within itself and of reflections. In other words, the group of uni-modular linear transformations (3.1) which corresponds to the stable leading motion, has a representation in the proper Lorentz group augmented by a reflection. The group of proper Lorentz transformations with the transformation of reflection represents the complete Lorentz group. The theorem has thus been proved.

(*Note*. The presented proof makes it possible to drop the hypothesis on the reducibility of the equations in variation of Poincaré if one requires to begin with the existence of a "sign-definite" (in the sense of Liapunov) invariant.

$$\sum c_{\mu\dot{\nu}}\xi^{\mu}\overline{\xi^{\nu}}$$

The stability of the leading motion will then follow directly from the existence of such an invariant on the basis of the general theorem of Liapunov on the stability of motion (*Obshchaia zadacha,* [*General problem*] p.61)).

4. Third Proof. One can give still a third proof without the use of the results of spinor analysis and without drawing upon the geometry of Lobachevskii.

Let us consider the equations in the variations of Poincaré for the leading motion

$$\frac{d\xi}{dt}=\frac{\partial^2 H}{\partial p\partial q}\xi+\frac{\partial^2 H}{\partial p^2}\eta, \qquad \frac{d\eta}{dt}=-\frac{\partial^2 H}{\partial p^2}\xi-\frac{\partial^2 H}{\partial q\partial p}\eta$$

If the disturbed motion is stable and the equations in the variations are reducible in the sense of Liapunov, then the characteristic equation of the reduced system will have a pair of pure imaginary roots. Hence, the "sign-definite" Liapunov function V for the equations in the variations of Poincaré will be a real quadratic form whose total time derivative will be zero.

Let Liapunov's function be given by the Hermitian form

$$V = \varphi_1 \xi\bar{\xi} + \psi_1 \xi\bar{\eta} + \psi_2 \bar{\xi}\eta + \varphi_2 \eta\bar{\eta}$$

satisfying the criterion of sign-definiteness ($\phi_1 \neq 0$) and

$$C = \begin{vmatrix} \varphi_1 & \psi_1 \\ \psi_2 & \varphi_2 \end{vmatrix} > 0 \tag{4.1}$$

The first total derivative with respect to time of Liapunov's function is identically zero. This yields

$$\begin{aligned} \frac{dV}{dt} = &\varphi_1{}'\xi\bar{\xi} + \varphi_1\left(\frac{\partial^2 H}{\partial p\partial q}\xi + \frac{\partial^2 H}{\partial p^2}\eta\right)\bar{\xi} + \varphi_1\xi\left(\frac{\partial^2 H}{\partial p\partial q}\bar{\xi} + \frac{\partial^2 H}{\partial p^2}\bar{\eta}\right) + \\ &+ \psi_1{}'\xi\bar{\eta} + \psi_1\left(\frac{\partial^2 H}{\partial p\partial q}\xi + \frac{\partial^2 H}{\partial p^2}\eta\right)\bar{\eta} + \psi_1\xi\left(-\frac{\partial^2 H}{\partial q^2}\bar{\xi} - \frac{\partial^2 H}{\partial q\partial p}\bar{\eta}\right) + \\ &+ \psi_2{}'\bar{\xi}\eta + \psi_2\left(\frac{\partial^2 H}{\partial p\partial q}\bar{\xi} + \frac{\partial^2 H}{\partial p^2}\bar{\eta}\right)\eta + \psi_2\bar{\xi}\left(-\frac{\partial^2 H}{\partial q^2}\xi - \frac{\partial^2 H}{\partial q\partial p}\eta\right) + \\ &+ \varphi_2{}'\eta\bar{\eta} + \varphi_2\left(-\frac{\partial^2 H}{\partial q^2}\xi - \frac{\partial^2 H}{\partial q\partial p}\eta\right)\bar{\eta} + \varphi_2\eta\left(-\frac{\partial^2 H}{\partial q^2}\bar{\xi} - \frac{\partial^2 H}{\partial q\partial p}\bar{\eta}\right) \equiv 0 \end{aligned}$$

Comparing the coefficients of $\xi\bar{\xi}$, $\xi\bar{\eta}$, $\bar{\xi}\eta$, $\eta\bar{\eta}$ we obtain the following system of linear differential equations

$$\begin{aligned} &\varphi_1{}' + 2\frac{\partial^2 H}{\partial p\partial q}\varphi_1 - \frac{\partial^2 H}{\partial q^2}(\psi_1 + \psi_2) = 0, \qquad \psi_1{}' + \frac{\partial^2 H}{\partial p^2}\varphi_1 + \frac{\partial^2 H}{\partial p\partial q}\psi_1 - \frac{\partial^2 H}{\partial q^2}\varphi_2 = 0 \\ &\varphi_2{}' - 2\frac{\partial^2 H}{\partial p\partial q}\varphi_2 + \frac{\partial^2 H}{\partial p^2}(\psi_1 + \psi_2) = 0, \qquad \psi_2{}' + \frac{\partial^2 H}{\partial p^2}\varphi_1 - \frac{\partial^2 H}{\partial p\partial q}\psi_2 - \frac{\partial^2 H}{\partial q^2}\varphi_2 = 0 \end{aligned}$$

Indeed, if the right hand sides of these equations were not zero, then it would be possible to select the initial values ξ_0, η_0 so that dV/dt would not vanish at the initial moment (which can be chosen arbitrarily). This, however, is not possible by hypothesis.

It is easily verified by direct computation that the expression

$$C = \begin{vmatrix} \varphi_1 & \psi_1 \\ \psi_2 & \varphi_2 \end{vmatrix} = \varphi_1\varphi_2 - \psi_1\psi_2$$

is an integral. Indeed, according to the differential equations for the functions ϕ_a, ψ_a, we have

$$\frac{dC}{dt} = \varphi_1'\varphi_2 + \varphi_1\varphi_2' - \psi_1'\psi_2 - \psi_1\psi_2' =$$
$$= \left[-2\frac{\partial^2 H}{\partial p\partial q}\varphi_1 + \frac{\partial^2 H}{\partial q^2}(\psi_1 + \psi_2)\right]\varphi_2 + \left[2\frac{\partial^2 H}{\partial p\partial q}\varphi_2 - \frac{\partial^2 H}{\partial p^2}(\psi_1 + \psi_2)\right]\varphi_1 +$$
$$+ \left[\frac{\partial^2 H}{\partial p^2}\varphi_1 + \frac{\partial^2 H}{\partial p\partial q}\psi_1 - \frac{\partial^2 H}{\partial q^2}\varphi_2\right]\psi_2 + \left[\frac{\partial^2 H}{\partial p^2}\varphi_1 - \frac{\partial^2 H}{\partial p\partial q}\psi_2 - \frac{\partial^2 H}{\partial q^2}\varphi_2\right]\psi_1 \equiv 0$$

The rest of this proof continues as the preceding one.

5. Fourth Proof. It is possible to find the solution in closed form for one of the primitive cases.

Let

$$2T = \sum a_{ij} p_i p_j, \qquad U = U(q_1, \ldots, q_k)$$

where T and U are the kinetic energy and the force function for some mechanical system with k degrees of freedom for which the $q_1, \ldots, q_k$ are the coordinates, while $p_1, \ldots, p_k$ are the momenta.

Let $V(t, q_1, \ldots, q_k, \alpha_1, \ldots, \alpha_k)$ be the complete Jacobi integral of the equation

$$\frac{\partial V}{\partial t} + H = 0$$

where $H = T - U$ is the Hamiltonian; a_{ij} does not depend explicitly on t.

Hence,

$$\frac{\partial V}{\partial t} = -h$$

where h is a constant force. Let us consider the function $\phi(V)$. We have

$$\varphi_{tt} = \varphi'' h^2$$

and

$$\frac{\partial}{\partial q_i}\left(a_{ij}\frac{\partial V}{\partial q_j}\right) = \frac{\partial}{\partial q_i}\left(\varphi' a_{ij}\frac{\partial V}{\partial q_j}\right) = \varphi'' a_{ij}\frac{\partial V}{\partial q_i}\frac{\partial V}{\partial q_j} + \varphi'\frac{\partial}{\partial q_i}\left(a_{ij}\frac{\partial V}{\partial q_j}\right)$$

whence,

$$L[\varphi] = \sum \frac{\partial}{\partial q_i}\left(a_{ij}\frac{\partial \varphi}{\partial q_j}\right) = \varphi'' \sum a_{ij} p_i p_j + \varphi' L[V]$$

But

$$\sum a_{ij} p_i p_j = 2(U + h)$$

When $h \neq 0$, we have

$$L[\varphi] = \varphi_{tt} \frac{2(U+h)}{h^2} + \varphi' L[V]$$

Now we will assume that the leading motion is stable for fixed a_1, $\ldots$, a_k. We have

$$\eta_j = \delta p_j = \sum \frac{\partial^2 V}{\partial q_j \partial q_i} \xi_j$$

The equations in the variations of Poincaré yield the following system

$$\frac{d\xi_i}{dt} = \sum \frac{\partial^2 H}{\partial p_i \partial q_j} \xi_j + \sum \frac{\partial^2 H}{\partial p_i \partial p_s} \eta_s = \sum \frac{\partial a_{is}}{\partial q_j} p_s \xi_j + \sum a_{is} \eta_s =$$
$$= \sum \frac{\partial a_{is}}{\partial q_j} \frac{\partial V}{\partial q_s} \xi_j + \sum a_{is} \frac{\partial^2 V}{\partial q_j \partial q} \xi_j = \sum \frac{\partial}{\partial q_j} \left(a_{is} \frac{\partial V}{\partial q_s} \right) \xi_j$$

For the reduced system (this is true for any regular system) the characteristic number of the expression

$$\exp \int \sum \frac{\partial}{\partial q_r} \left(a_{rj} \frac{\partial V}{\partial q_j} \right) dt = \exp \int L[V]\, dt$$

must be zero if the motion is stable. For the primitive case this condition of stability is

$$L[V] = 0$$

(In the general case this condition is sufficient in order that one condition of stability be satisfied). In accord with the last relation we have

$$\rho \varphi_{tt} = L[\varphi] \quad \left(\rho = \frac{2(U+h)}{h^2} \right)$$

This equation is of the hyperbolic type because $2T = \Sigma a_{ij} p_i p_j$ is a positive definite quadratic form.

In the general case

$$\int L[V]\, dt \quad \text{is a bounded function}$$

6. Development of the optical-mechanical analogue. In order to show that the obtained result represents a development of the optical-mechanical analogue, it is sufficient to note that the equations in the variations of Poincaré were used only for the sake of convenience in the application of the general results of the theory of stability of motion of Liapunov. One can start out directly with the fundamental propositions of the optical-mechanical analogue of Hamilton.

Indeed, the function of action of Hamilton V satisfies the fundamental relation of the contact transformations between the values of the coordinate q and the momentum p for the instant t and for the initial instant t_0

$$\delta V = p\delta q - p^\circ \delta q^\circ$$

The outer product or bilinear covariant of this expression

$$[\delta p, \delta q] = [\delta p^\circ, \delta q^\circ] \tag{5.1}$$

represents nothing more than the Poincaré invariant for the equations in the variations. Therefore, if one now demands stability of the deviations δq, δp for small initial deviations of the coordinate δq^0 and of the momentum δp^0,. then the bilinear covariant of the fundamental relation of the optical-mechanical analogue of Hamilton (5.1) yields the representation of the transformations arising hereby of δp, δq in the fundamental group of the post-Huygens theory of light, i.e. the complete Lorentz group.

An essential addition to the results of Hamilton was made by Jacobi in his well known theorem on the properties of the complete integral $V(t, q, \alpha)$ of the partial differential equation

$$\frac{\partial V}{\partial t} + H\left(t, q, \frac{\partial V}{\partial q}\right) = 0$$

Jacobi established the relations

$$p = \frac{\partial V}{\partial q}, \qquad -\beta = \frac{\partial V}{\partial \alpha}$$

where α, β are constants. From these equations we obtain

$$\delta V = p\delta q - \beta\delta\alpha$$

or, forming the bilinear covariant, we have

$$[\delta p, \delta q] = [\delta\beta, \delta\alpha]$$

The general invariant of dynamics makes it possible to show how the derived results depend on small deviations of the parameters α and β.

If one has a mechanical system with k degrees of freedom, and if furthermore, all variables can be completely separated, then the presented development of the optical-mechanical analogue can be extended to such a system directly in the form of a k-multiple representation. The general case is of interest for its own sake in analytical mechanics.

6. Problems. Several interesting problems arise in connection with the solution of Klein's problem, some of which were solved quite long ago.

It seems that Lagrange was the first to consider properties of mechanical systems admitting an invariance of relations under possible transformations of the elementary group of translation and rotation of a solid body. He did this in his work "Analytical Mechanics". A.P. Kotel'nikov considered the general transformation of this group. For holonomic relations of the group of possible displacements which are connected with invariants, Poincaré gave a general treatment.

A second problem that has been considered deals with mechanical systems that have certain transformations which leave invariant certain relations and the work or force function or the total energy of the system. Lagrange considered elementary transformations of the group of motions of a rigid body which leave invariant certain relations among given forces and the work done by these forces. I have treated the transformations of the general group of all possible displacements which leaves Lagrange's function invariant.

The group of real motions has also been studied. Lagrange established the properties of mechanical systems whose real displacements are within the group of possible displacements. Hamilton showed that the general transformations of Lagrange's coordinates of the canonical system q_s and of the conjugate momenta p_s represent a group of contact transformations. The general group of contact transformations has been well studied. In this work the group of disturbed motions of a stable, leading motion of a canonical system has been determined.

Other natural problems of dynamics, which are connected with the group of possible displacement, with the group of real motion, with groups that leave invariant fundamental mechanical functions, or problems related to representations of these groups, still await solution.

Translated by H.P.T.